An Improved Architecture for Implementing Narrow Bandwidth Low Pass Recursive Filters

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ABSTRACT

The poles of a recursive digital filter shift their position when the denominator polynomial coefficients are quantized to a fixed bit width approximation. The size of the root migration due to coefficient quantization increases with filter order and is more severe with reduced bandwidth. A sufficient number of bits must be allocated to the quantized coefficients to preserve the spectral fidelity of the filter design. Narrowband low pass filters also exhibit large numerical gain which leads to extended bit width requirements for internal registers and multipliers. We present a technique to implement high order very low-bandwidth recursive lowpass filters without the brute force requirement for extended precision coefficients and registers.

1. INTRODUCTION

Within a scale factor, a recursive digital filter is defined by its poles and zeros, the denominator and numerator roots of its transfer function. The roots are often presented graphically as in figure 1. The denominator and numerator polynomials formed by expanding the factored form are shown in (1). Without quantization, the two denominators of (1) are identical. With coefficient quantization, the coefficients (a_m) are replaced with approximate coefficients ($a_m+\Delta a_m$), which causes the roots to move from (p_m) to ($p_m+\Delta p_m$).

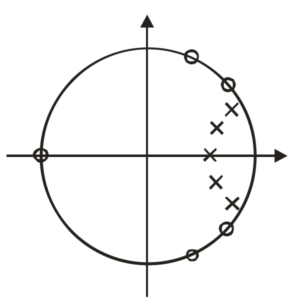

Figure 1. Pole-Zero Diagram of 5-th Order IIR Filter

$$H(Z) = \frac{\prod_{k=1}^{N}(Z - z_k)}{\prod_{k=1}^{N}(Z - p_k)} = \frac{\sum_{n=1}^{N} b_n Z^{N-n}}{\sum_{n=1}^{N} a_n Z^{N-n}} \qquad (1)$$

We estimate the change Δp_m in root p_m due to the change Δa_m of the coefficient a_m using sensitivity analysis. Equation 2 shows the sensitivity coefficient.

$$S_{a_k}^{P_m} \doteq \frac{\Delta p_m}{\Delta a_k} = -\frac{(P_m)^k}{\prod_{\substack{n=1 \\ n\neq m}}^{N}(P_m - P_n)} \qquad (2)$$

To first order, the polynomial root P_m moves the reciprocal of the denominator of (2) which is seen to be the product of the distances between the root P_m and the remaining roots P_n of the polynomial. Thus if we have 5 roots in the same polynomial, and the distance from a selected root to each of his four companions is on the order of 10^{-2}, the expected root shift is on the order of 10^{+8} times the change in the coefficient. Root locus considerations suggest that the roots move radially out from their center of gravity, thus, almost assuredly, at least one root will move outside the unit circle when we quantize the denominator coefficients of a high order filter. It is for this reason we avoid designs with multiple roots in the same polynomial. Significantly reduced coefficient sensitivity is obtained by unpacking the polynomial to form a cascade of first and second order filters as shown in figure 3. A sensitivity of 10^2 is manageable for the roots of a second order polynomial separated by 10^{-2}.

In a similar vein, the gain between input and internal states of an IIR low-pass filter as shown in (3), is seen to be the inverse of the product of the distance from each pole in the polynomial to the Z=1 test point on the unit circle. For the 5-pole example, this gain is seen to on the order of 10^{10} which represents approximately 33 bits of word growth.

$$H(0)\Big|_{\text{Poles}} = \frac{1}{\prod_{k=1}^{N}(Z - p_k)}\Bigg|_{Z=1} = \frac{1}{\prod_{k=1}^{N}(1 - p_k)} \qquad (3)$$

To control the undesired attributes of an IIR filter, coefficient sensitivity and large processing gain, we unpack the denominator polynomial and implement IIR filters as a cascade of first and second order sub-filters with gain scaling between stages. This unpacking is shown in figure 2.

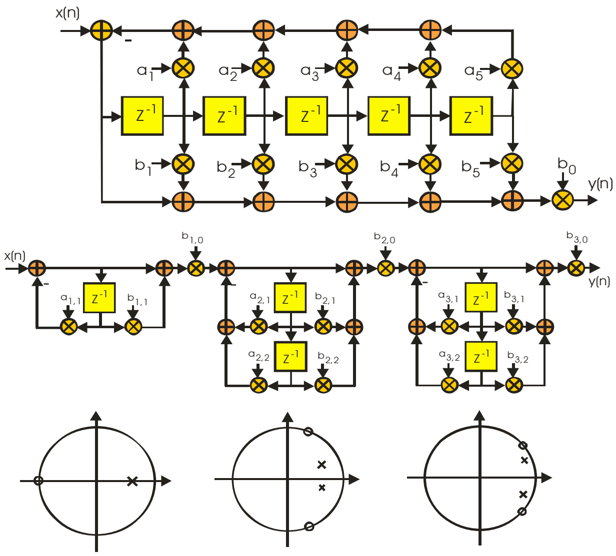

Figure 2. Unpacking A Single Stage 5-th Order Filter into a Cascade of First and Second Order Filters.

We now examine the sensitivity of root locations of a second order polynomial to quantization of its two coefficients. In (4) we show the polynomial formed from its factored form with roots x ± jy.

$$\begin{aligned} p(z) &= (z-x-jy)(z-x+jy) \\ &= z^2+(-2x)z+(x^2+y^2) \end{aligned} \tag{4}$$

Equation 5 shows us the standard representation of a second order polynomial.

$$p(z) = z^2 + a_1 z + a_2 \tag{5}$$

Equating corresponding terms from (4) and (5) we determine how the roots of the polynomial are related to its coefficients. This relationship is shown in (6).

$$\begin{aligned} &\text{Real part of root:} && x = -\frac{a_1}{2} \\ &\text{Distance from origin:} && R=\sqrt{x^2+y^2}=\sqrt{a_2} \end{aligned} \tag{6}$$

The relationship of (6) describes the locus of the root locations directly from the coefficients. Examining the locus offers insight into the coefficient sensitivity. Referring to figure 3, we see that the roots lie at the intersection of the line x = -a_1/2 and the arc of radius $\sqrt{a_2}$. We can see that when the roots are very close, the distance -a_1/2, and the radii $\sqrt{a_2}$ are nearly the same size and that the arc and the line are separated by a small angle and in the perspective of linear algebra represent an ill conditioned geometry. The intersection of the line and arc for low bandwidth filters will be difficult to control when the coefficients a_1 and a_2 are quantized. This is demonstrated in figure 4 which identifies the possible root locations due to quantized line-arc intersections for 8-bit a_1 and a_2 coefficients. Notice the sparseness of roots in the region near Z=1, corresponding to low bandwidth low pass filters.

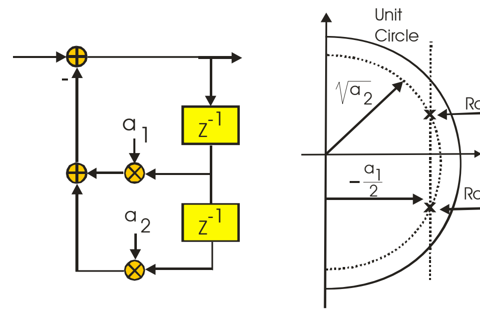

Figure 3. Locus of Roots, Second Order Polynomial with Real Coefficients

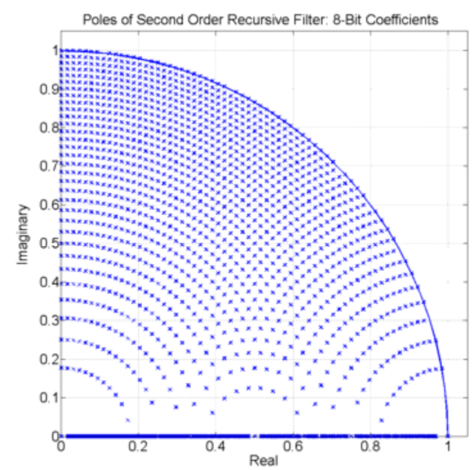

Figure 4. Possible Roots of Second Order Polynomial with 8-bit Quantized Real Coefficients.

The traditional response to the requirement to place roots in this sparse region is to reduce the size of the region by increasing the number of bits used to represent the filter coefficients. Another approach is to limit filters to a cascade of first order polynomials. This requires the use of complex coefficients to realize complex roots. The structure of the single complex pole filter, recognized as the normal filter, is shown in figure 5.

The root location for a first order polynomial is the negative of the complex coefficient. Since the real and imaginary components of the coefficient lie on orthogonal Cartesian coordinates, the quantized root grid coincides with the same grid. This is demonstrated in figure 6 which shows the root locations for 7-bit Real and Imaginary coefficients. Notice the uniformity of root distribution enable filters in the region corresponding to low bandwidth low pass filters. The penalty we pay for using the complex coefficient filter is that it takes 4-multiplies to form the single pole. Incidentally, the conjugate pole does not have to be formed in a second filter since we can obtain the real output sequence as the scaled imaginary component, via its residue, of the single pole filter. Thus the complex feedback coefficient forms both roots at a cost of 2-multiplies per root.

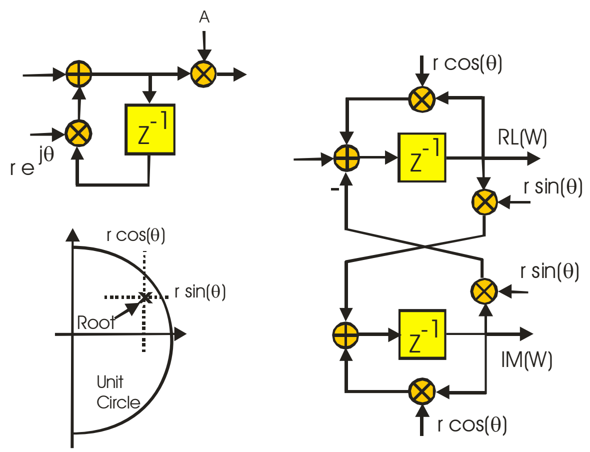

Figure 5. First-Order Complex Root from First Order Polynomial with Complex Coefficient

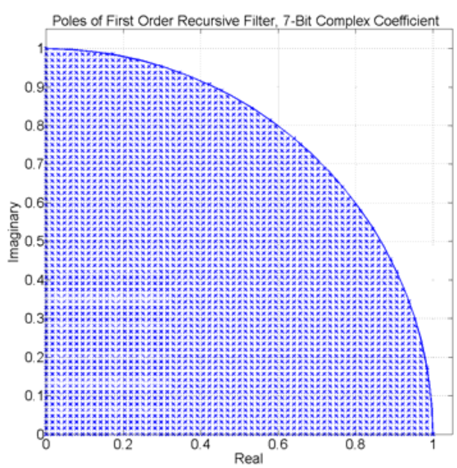

Figure 6. Possible Roots of First Order Polynomial with 7-bit Quantized Complex Coefficient.

2. ALTERNATE FILTER ARCHITECTURE

Rather than change the filter structure to the normal form to enable access to a uniform Cartesian grid in the formerly sparse region near DC we propose an alternate option which relocates the roots to the region near the quarter sample rate which already exhibits a Cartesian grid. In this structure, rather than design a low pass filter with the desired bandwidth at DC, we design a real narrowband filter with the same bandwidth centered at the ± quarter sample rate. We up convert the input spectra from DC to the quarter sample rate, filter the complex series with a pair of these filters, and then down convert the filtered series back to baseband. This structure is shown in figure 7 for the very efficient two-path polyphase all-pass filter. Here 5-multiplies form 5-poles and 5-zeros and again in the band pass 10-poles and 10-zeros.

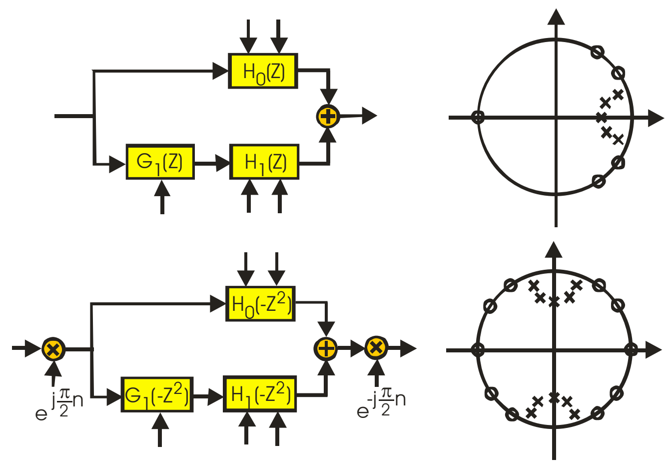

Figure 7. Block Diagram and Root Location for Low Pass Prototype and Quarter Sample Rate Band Pass Filters

The first thing we note is that the up and down conversions, by the complex terms exp(±j n π/2), at the input and output of the band pass filter are trivial and at most are performed with sign changes and zero insertions. Note that the real part and imaginary parts of the input product is performed by the repeated sequences [1 0 -1 0] and [0 1 0 -1] respectively. This has an interesting interaction with the band pass filter which we now address.

Examining the filter, we note that the filter structure for the low pass filter is the same as for the band pass filter. The filters differ primarily in zero-packing, replacing the polynomial in Z by a polynomial in $-Z^2$; a special case of the low-pass to band-pass transformation. The denominators for the low pass and band pass filters shown in figure 7 are shown in table 1.

Table 1. Denominators for Low Pass and Band Pass Filters

$H_0(Z)$	Z^2 -1.941047 Z + 0.942547
$G_1(Z)$	Z - 0.961481
$H_1(Z)$	Z^2 -1.978362 Z + 0.979889
$H_0(-Z^2)$	Z^4 + 1.882559 Z^2 + 0.888380
$G_1(-Z^2)$	Z^2 + 0.924390
$H_1(-Z^2)$	Z^4 + 1.954166 Z^2 + 0.960208

Since the polynomial has been zero packed, the filter registers present two delays between successive filter taps. Examining the two series formed by the input cosine sine heterodynes we note the non zero samples of the two series presented to the In-phase and Quadrature-phase versions of the filter are offset by one sample. Since the filter registers are two samples wide, one filter register at a given instant contains the sequence [0 -d(n-2) 0 d(n)] while the other contains the sequence [-d(n-3) 0 d(n-1) 0]. We come to the conclusion that the two filter registers can be interleaved so that only a single filter is required to process the I-Q data formed by the input up-converter. The input to this filter is seen to be the interleaved sequence [I(n), j*Q(n+1), –I(n+2), –j*Q(n+2), ….]. The output heterodyne that basebands the filter output accesses the real and imaginary output samples from successive samples of the single filter. This interleaving and de-interleaving at the input and output of the quarter sample frequency band pass filter is shown in figure 8.

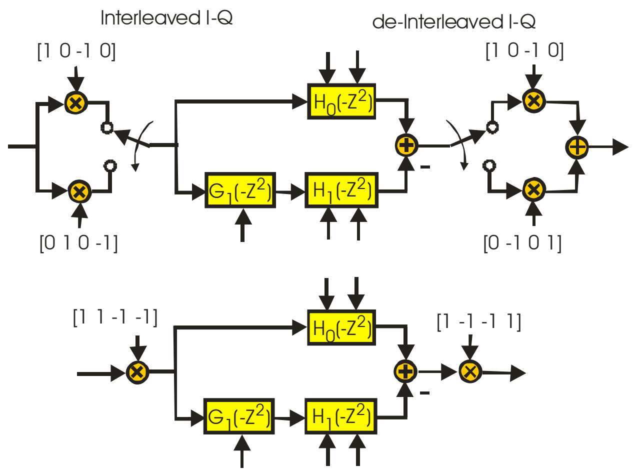

Figure 8. Interleaving at Input and Output of Single Quarter-Sample Rate Band Pass Filter.

3. COEFFICIENT SENSITIVITY of ALTERNATE ARCHITECTURE

Figure 9 presents the pole-zero diagrams of the prototype low pass filter and of the quarter sample rate band pass filter. Also shown is the spectral response of the two filters exclusive of the input and output heterodynes associated with the band pass filter. The heterodyned version of the band pass filter will be seen to have a spectral component at DC and one at fs/2. The output of the heterodyned filter will require a low pass filter with wide transition band to suppress the fs/2 band. This filter will be seen to only require one multiply. We will examine the time and frequency response of the system shortly.

We now continue with the verification that the alternate architecture is significantly less sensitive to coefficient quantization than is the prototype low pass filter. Figure 10 shows the root migration of the second order polynomials describing the low pass prototype filter and of the quarter sample rate band pass filter for a range of coefficient quantization spanning 24 bits to 10 bits. A root of the low pass prototype filter migrated to the unit circle at 10-bit quantization hence was not included in the figure. Note the remarkably small amount of root migration for the quarter sample rate band pass filter compared to that of the low pass prototype. Similar behavior is observed for quantized coefficients of the standard bi-quadratic form of the low pass and band pass filter structures.

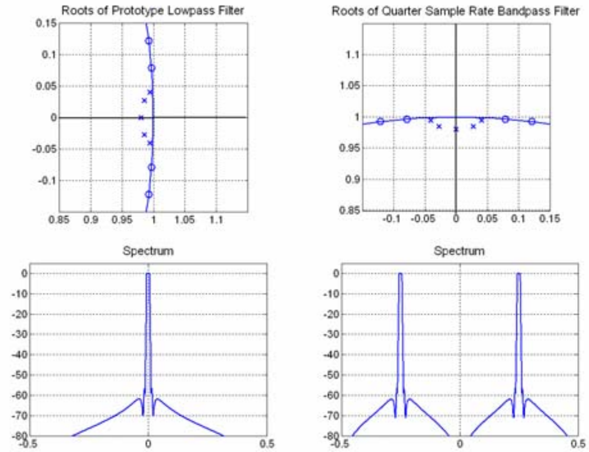

Figure 9. Root Locations of Low pass Filter and of Quarter-Sample Rate Band Pass Filter

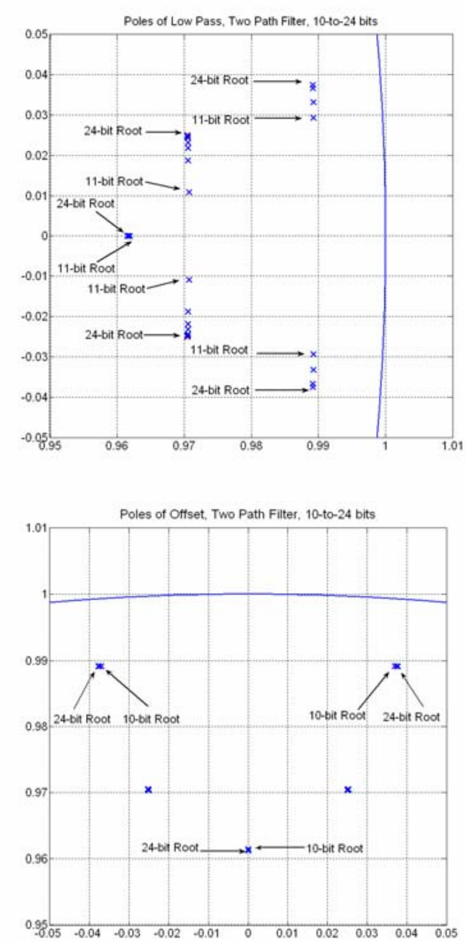

Figure 10. Roots of Low Pass Prototype and Quarter Sample Rate Band Pass Filter, Second Order Polynomial Coefficients Quantized 24-Bits to 10-Bits

Figure 11 shows the impulse response of the In-phase and Quadrature components of the up-converted time series as well as the down-converted output time series and its spectral response. Note the replica spectral response at the half sample rate which is still to be suppressed by a clean-up low pass filter. Consistent with the replica spectrum, the real impulse response of the filter has alternate sample values equal to zero as if it had been zero-packed.

Figure 12 shows the impulse response of a prototype low pass filter, of the down converted band pass filter, and of the low pass filtered version of the down converted filter. Also shown are their corresponding spectra.

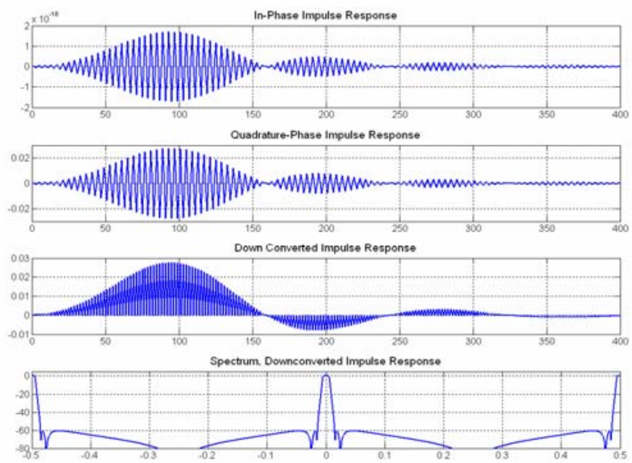

Figure 11. Impulse Response of the In-Phase and Quadrature-Phase Components of the Up-Converted Time Series and the Down-Converted Output Time Series and Spectrum

4. REVIEW AND CONCLUSIONS

We have reviewed the difficulty we encounter when operating a recursive filter at very high ratios of sample rate to bandwidth. The problem is due to shift of pole positions in response to coefficient quantization. The shift is greater when there are a large number of roots in the filter polynomial. Good design practices have us decouple roots by placing them in different polynomials implemented in distinct sub-filters. Similar considerations show that the numerical gain of a filter is proportional to the ratio of sample rate to filter bandwidth. This gain had to be scaled out of the filter and good design strategies dictate that the scaling be distributed over multiple small filters to avoid having to deal with very wide words in the processing stream.

We reviewed the finite arithmetic effects of first and second order sub-filters. We reminded the reader that the interaction between the two coefficients of a second order polynomial leads to an ill conditioned coupling when the recursive filter is designed for low bandwidth near zero frequency. The conditioning is worse for closely spaced poles near DC and the conditioning is best for widely spaced poles centered at the quarter sample rate.

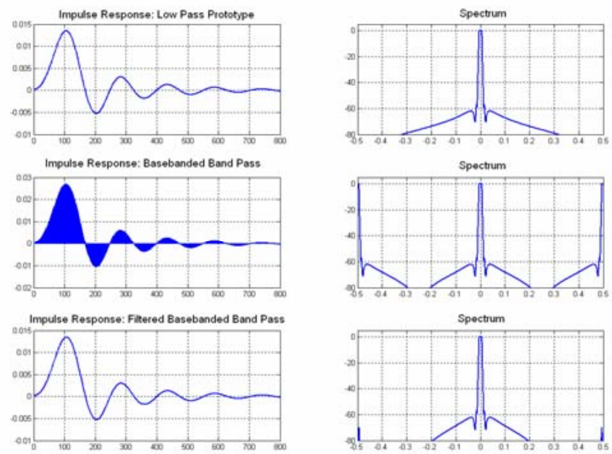

Figure 12. Impulse Responses; Prototype Low Pass, Down Converted Band Pass, and Low Pass Filtered Version of the Down Converted Filter and Associated Spectra.

Following this lead, we designed real coefficient band pass filters with the desired bandwidth centered at the quarter sample rate. We noted that the translated filter has essentially the same structure as the low pass filter except that the polynomials are zero packed with each Z replaced with $-Z^2$, a default version of the low-pass to band pass transformation. To access the bandwidth of the offset filter we heterodyned the input signal to the quarter sample rate. We observed that the interleaved I and Q components of the up converted time series meshed perfectly with zero-packed impulse response and register structure of the band pass filter so that the I and Q components could be interleaved in the same filter. Thus there is no second filter required to process the complex up-converted time series! This is a very efficient method of implementing narrow bandwidth low pass filters. This method requires a clean up filter to finish the process.

5. REFERENCES

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