Flexibility, Performance and Implementation Advantages of Recursive, Linear and Non Linear Phase, Polyphase Filters in Transmitters and Receivers

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ABSTRACT

We present and compare procedures that insert recursive (IIR) filters in processing tasks that perform filtering with embedded resampling. Non-Recursive (FIR) filters are traditionally used in resampling filters due partly to the ease with which a FIR filter can be partitioned into poly-phase sub-filters. Recursive filters formed from M-path all-pass filters can be reconfigured to perform these same embedded resampling tasks. We demonstrate the performance of M-stage polyphase partitions of the IIR filters exhibiting uniform and non-uniform phase.

1. INTRODUCTION

A prototype low-pass FIR filter is often partitioned into one or more polyphase filters to efficiently perform filtering and resampling in a single process. The simultaneous resampling and filtering operation can also be implemented by an IIR filter. We demonstrate the structure of simple recursive all-pass filters that can be inserted as sub-filters in polyphase resampling filters. As a vehicle to present this material, we define a processing task and then present and compare a number of conventional and unconventional filter architectures that perform the task.

The filtering task we address is that of reducing the bandwidth of an input data stream with a filter satisfying a specified performance and then reducing the sample rate by an amount commensurate with the reduction in bandwidth. The signal parameters and the filter specifications are shown in figure-1. The filter processes input data sampled at 640 kHz, reduces the bandwidth to a nominal baseband bandwidth of ± 20 kHz and resamples the output data 8-to-1 to obtain an 80 kHz output sample rate.
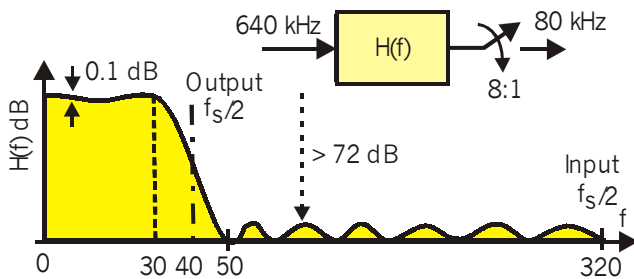

Figure 1. Frequency Response of Filter for Filter and Resample Processing Task

The filtering and resampling tasks can be decoupled by performing the filtering, computing one output for each input then down sample, keeping one output sample in eight and discarding the remaining seven. The total workload for the eight samples is assigned to the retained sample, which significantly raises the computational cost for the retained sample. This is the motivation for seeking alternate processing schemes that by embedding the resampling process in the filter commits computation resources only to the retained samples.

Three filter structures that satisfy the filter specifications and decouple the filtering task from the resampling task are shown in figure-2. These are a 7-th order elliptic filter, a 104-tap FIR filter, and a 9-th order two-path all-pass recursive filter. We use these filters to establish a reference workload levels as well as a simple introduction to the recursive all-pass structure.
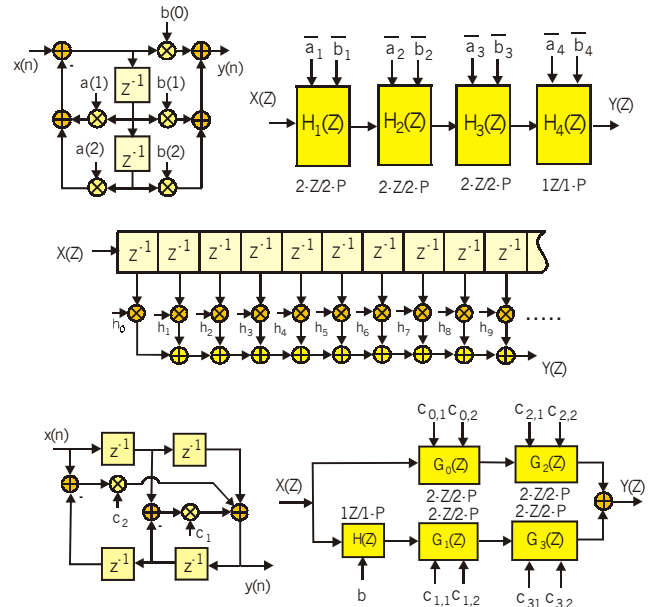

Figure 2. Seventh Order Elliptic Filter, 105-Tap FIR Filter, and Ninth Order Polyphase All-Pass Filter.

The standard recursive filter is formed from a set of bi-quadratic sections that require one multiply per pole, one multiply per zero, and one scale factor per segment. The 7-th order elliptic filter requires 18 ops per input point, which translates after the 8-to-1 down sample to 144

ops per retained output point. Here an "op" refers approximately to a multiply, an add, and the appropriate data and coefficient fetches in the process.

The standard FIR filter is formed as a set of multiply-accumulate operators. The 104-tap FIR filter requires 104 ops per output point and since the filter is non-recursive we only compute the retained output samples. The workload per output is amortized over 8-inputs, so that the FIR filter requires 104/8 or 13 ops per input.

First and second order all-pass segments in which each pole and reciprocal zero is formed by single filter coefficients form the two-path all-pass filter. The phase response of the two unity-gain paths have the same phase shift in the pass-band and differ by approximately π in the stop band. When the outputs of the two paths are added, the signal experiences destructive interference in the stop band Thus the zeros of the all-pass sections, formed while building the section poles, becomes the filter stop-band zeros by the simple expediency of adding the two paths. These filters require less than half the coefficients of a standard bi-quadratic implementation. The phase response of the two filter paths and the spectral gain of the filter are shown in figure-3. The 9-th order two-path filter requires 9 ops per input point, which translates after the 8-to-1 down sample to 72 ops per retained output point.
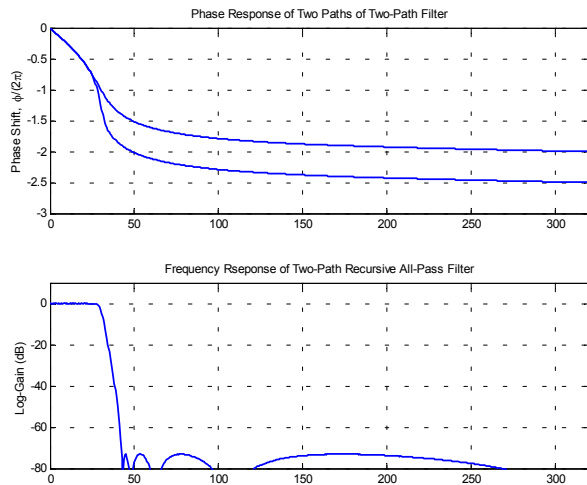

Figure-3 Phase Response of Two-Paths of Two-Path Recursive All-Pass Filter and Frequency Response for Sum of Two Paths

2. POLYPHASE RE-SAMPLING FIR FILTERS

The 108-tap FIR filter presented in the previous section can be partitioned into a polyphase filter that permits the interchange of the filtering operation and the resampling operation. This is accomplished by describing the FIR filter as a two-dimensional filter as shown in equations 1 through 3 for the general case of an M-to-1 down sampler.

Here the double sum is an inner sum over sub-filters containing data and coefficients separated by multiples of M input samples and then the outer sum over the sub-filter outputs. This structure is shown in figure-4.

$$H(Z) = \sum_{n=0}^{N-1} h(n)Z^{-n} \tag{1}$$

$$H(Z) = \sum_{r=0}^{M-1} \sum_{n=0}^{(N/M)-1} h(r+nM)Z^{-(r+nM)} \tag{2}$$

$$= \sum_{r=0}^{M-1} Z^{-r} \sum_{n=0}^{(N/M)-1} h(r+nM)Z^{-nM}$$
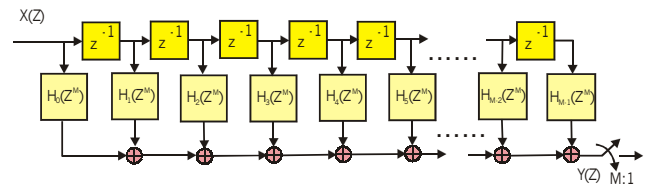
$$= \sum_{r=0}^{M-1} Z^{-r} H_r(Z^M) \tag{3}$$

Figure-4 Polyphase Partition of FIR Filter: A Two Dimensional Sum over Sub-Filters Processing Samples Starting on Index r and Incrementing by M (Z^{-M}).

The filter is followed by the M-to-1 down sampler so that an output is required once for every M-inputs. Since the M-units of delay at the input rate in the sub filters is a single delay at the output rate, we can interchange the order of the down sampler and the filter stage. When doing so we must replace the Z^{-M} at the input rate with Z^{-1} at the output rate. The interaction of the tapped delay line and the re-samplers at the input to the sub filters can be seen to be equivalent to an input commutator delivering the current sample to stage "0", a sample one-unit old to stage "1", a sample two-units old to stage "2" and so on. This interpretation is shown in figure-5.

In a polyphase partitioned and re-sampled filter, the workload per input sample is simple the workload of each polyphase segment N/M which for our example is 104/8 or 13 ops/input. For the record, N/M is a constant for each filtering task and is determined by the ratio of output sample rate to filter transition bandwidth as well as in-band and out-of-band ripple levels.

The phase shift of the separate paths in the partitioned but not yet re-sampled M-path filter are shown in figure-6. We see that in the zero-th Nyquist zone of the output sample rate spectral region, the paths have the same phase shift but in successive Nyquist zones the path phase shifts differ by 2π/8, then 2(2π/8), then again 3(2π/8) and so on. Since the spectral regions in the higher Nyquist zones differ by multiples of 2π/8, the 8 roots of unity, their components destructively cancel when summed. Spectra residing in the
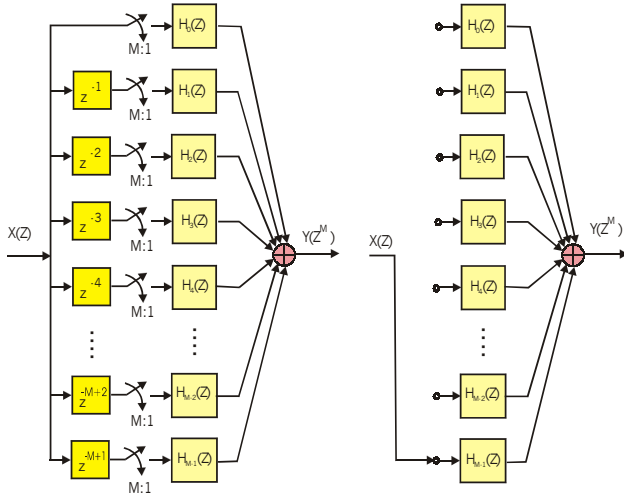
Figure-5 Polyphase Partition: Commutated M-Path Filter

higher order Nyquist zones preserve their relative phase when aliased to the baseband as a result of the 8-to-1 down sampling. Thus when summed, the aliased spectral terms are destructively cancelled effectively achieving the filtering task after rather than prior to down sampling.
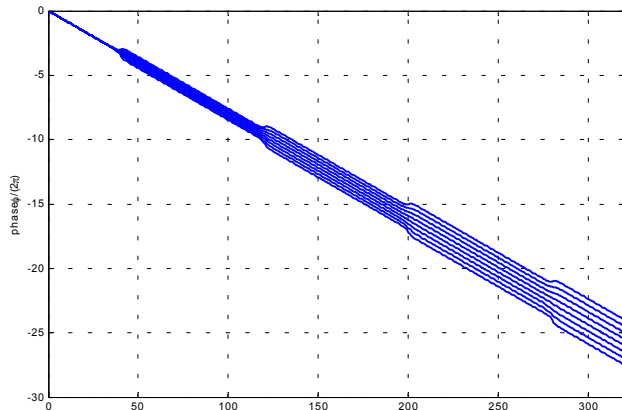

Figure-6 Phase Response of Separate Paths of Polyphase Partitioned, but not Re-Sampled Eight Path Filter

3. POLYPHASE RE-SAMPLING IIR FILTERS

The polyphase partition of a FIR filter, as shown in equation-1, formed filter segments describable as polynomials in Z^M. It is not generally possible to obtain similar partitions of standard recursive filters such as Butterworth or Elliptic filters. We might then, erroneously conclude that we cannot form polyphase IIR filters that will support the re-ordering of filtering and resampling. In fact we can form such filters, but they bear little resemblance to standard IIR filter structures. These structures start with the description shown in equation-3. The arms of the polyphase IIR filter must be all-pass filters formed as polyno-

mials in Z^M. In fact they are formed by a cascade of first and second order all-pass filters in Z^M as shown in equation-4 and in figure 7. Note that each coefficient in this filter structure is responsible for M-poles and M-zeros residing on line segment matching the M-roots of unity.

$$H(Z^M) = \frac{1 + a_1 Z^M}{Z^M + a_1}$$

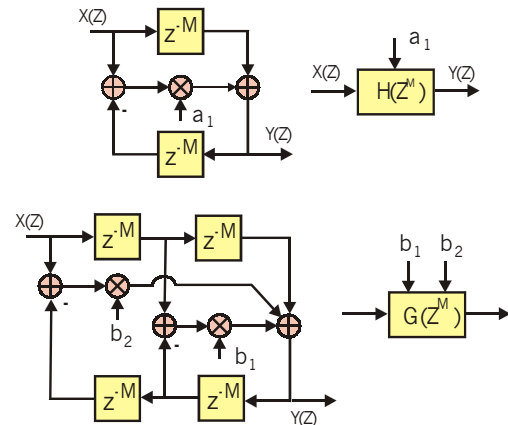$$G(Z^M) = \frac{1 + b_1 Z^M + b_2 Z^{2M}}{Z^{2M} + b_1 Z^M + b_2}$$

(4)

Figure-7 First and Second Order All-Pass Filters in Z^M

A design program written by our group at SDSU computed a set of weights for the 8-path recursive all-pass filters. All the filter segments were first order all-pass filters in Z^{-8}, with the first 5-arms requiring 3-stages each and the remaining 3-arms requiring 2-stages each. Thus the total filter required 21 weights. When used as a re-sampled polyphase filter the workload is 3 ops or 2 ops per input point with an average of approximately 2.7 ops per input.
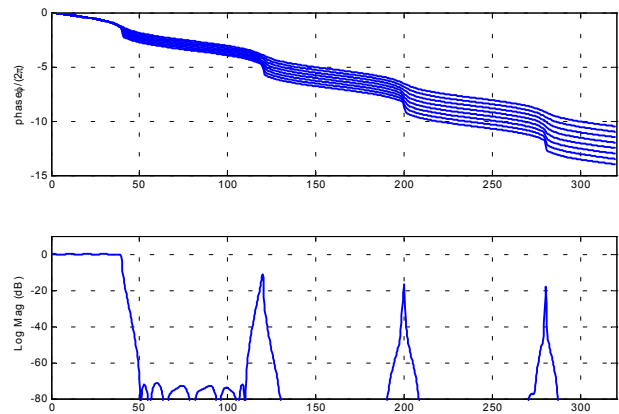

Figure–8 Phase Response of 8-paths and Frequency Response of 8-Path Recursive Polyphase Filter

The 2.7 ops/input is a significant reduction relative to the 13 ops/input for the polyphase FIR filter.

The separate phase responses of the 8-path filter and the frequency response of the composite filter are shown in figure 8. As with the FIR partition, the phases match in the first Nyquist zone, but differ by multiples of $2\pi/8$ in higher Nyquist zones. A significant difference between the recursive and the non-recursive phase structures is the transition between Nyquist zones. In the FIR, the phase in the transition regions both increase and decrease to sustain the destructive cancellation while in the IIR, the phase in the transition regions only decrease and does not sustain the destructive cancellation. In a sense, the filter exhibits transition regions between successive stop-band regions. These transition regions are of little concern since they exactly overlap the pass band to stop band transition region when the spectra aliases due to down sampling.

A second design program written by our group at SDSU computed a set of weights for a linear phase version of the 8-path recursive all-pass filters. In this option, the top path is implemented as a delay line, which of course exhibits linear phase. The remaining filter paths form an equal ripple approximation to the phase of the first path so that the phase obtained in the first Nyquist zone exhibits equal ripple response. The filter segments in the remaining arms required to meet the filter specifications were found to be two first order and two second order all-pass filters in Z^{-8}. This form of filter required a total of 42 weights and when used as a re-sampled polyphase filter the workload is 42/8 or approximately 5.3 ops per input.

The separate phase responses of the linear phase 8-path filter and the frequency response of the composite filter are shown in figure 9. As with the previous IIR partition, the phases match in the first Nyquist zone, differ by multiples of $2\pi/8$ in higher Nyquist zones and exhibit transition regions between the successive stop band regions.
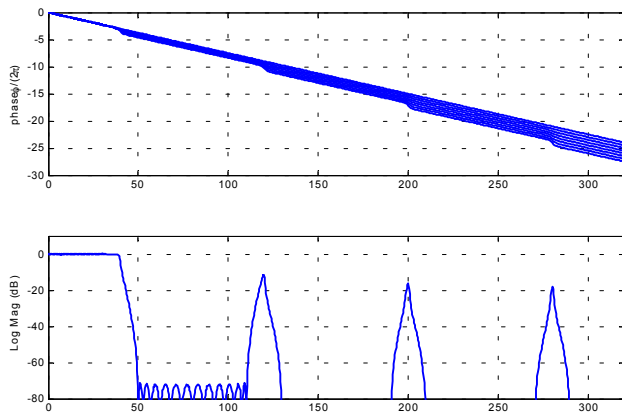

Figure 9. Phase Response of Linear-Phase 8-paths and Frequency Response of 8-Path Recursive Polyphase Filter

4. DYADIC HALF BAND RE-SAMPLING FILTERS

Another popular method of filtering with embedded resampling uses a sequence of successive filter and down sampling stages. In particular, a succession of half band filters with 2-to-1 down sampling proves to be a particularly efficient resampling scheme. The advantage is related to the fact that early filter stages operating at the high input rate tend to be of low order while later stages operating at the reduced output rate tend to be of high order. The required spectral responses of three successive half-band filters is shown in figure 10 where we can easily see the reduction in transition bandwidth in successive filters.
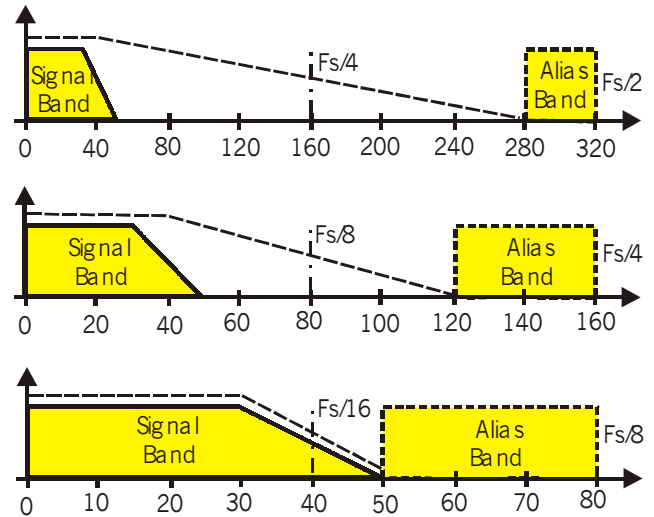

Figure 10. Spectral Characteristic of Three Successive Half Band Filters along with Signal and Alias Bands

We now examine the performance of various half-band resampling filters that meet the requirements suggested in figure 10. For ease of presentation, we will examine the filter spectra at a common input sample rate. This is accomplished by zero packing the impulse responses of the second and third filter 1-to-2 and 1-to-4 respectively. A trivial way to realize the zero packing is to replace Z^{-1} with Z^{-2} in the second filter and to replace Z^{-1} with Z^{-4} in the third filter, a transformation shown in figure 11.
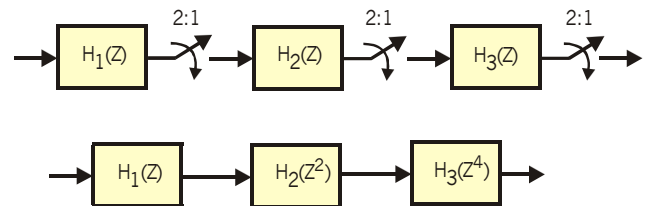

Figure 11. Cascade of Three Half-band Filters with 2-to-1 Down Samplers and Cascade of Same Filters Operating at Common Input rate Without Down Sampling

Three half-band FIR filters were designed so that the composite filter satisfied the filtering specifications shown in figure 1. The required filter lengths were found to be 8, 12, and 18 taps respectively. We compute the workload per input data point by processing the 8-input points required for 1-output and then amortize the work over the 8 inputs. When used in a polyphase 2-to-1 down sampling mode, the first filter requires 4 ops/input for 8-inputs, the second requires 6-ops/input for 4-inputs, and the third stage requires 9-ops/input for 2-inputs for a total workload of (4*8 + 6*4 + 9*2) or 74 ops/output or 9.25 ops/input. The spectral response of the FIR filter Dyad is shown in figure 12.

Three Half-Band FIR Filters: 8, 12, and 18 Taps, 9.25 Ops/Output
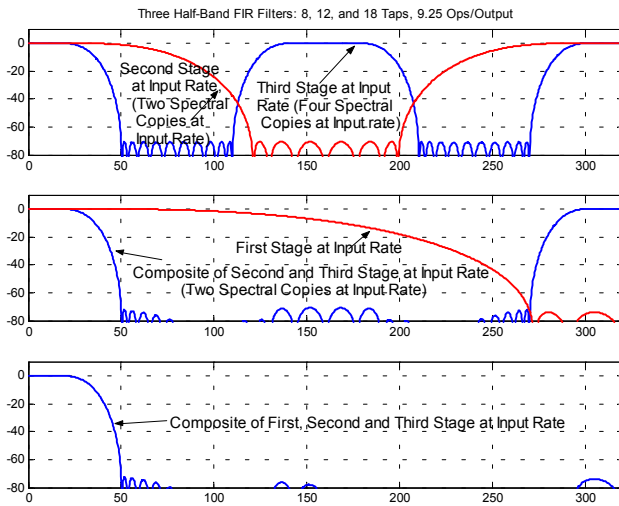
Figure 12. Spectral Response of Cascade Half-Band FIR Filters Referenced to Input Sample Rate

Three half-band recursive non-uniform phase all-pass filters were designed so that their composite spectral response met the filtering specifications. The required number of all-pass stages was found to be 2, 3, and 4 respectively. When used in a polyphase 2-to-1 down sampling mode, the first filter requires 1 ops/input for 8-inputs, the second requires 1.5-ops/input for 4-inputs, and the third stage requires 2-ops/input for 2-inputs for a total workload of (1*8 + 1.5*4 + 2*2) or 18 ops/output or 2.25 ops/input. Not bad! The spectral response of the IIR filter Dyad is shown in figure 13.

The half-band recursive all-pass filters with uniform phase were also to meet the filtering specifications. The required number of all-pass stages was found to be 2 first order, 1-first order with 1-second order, and 1-first order with 3-second order filters respectively. When used in a polyphase 2-to-1 down sampling mode, the first filter requires 1 ops/input for 8-inputs, the second requires 1.5-ops/input for 4-inputs, and the third stage requires 2.5-ops/input for 2-inputs for a total workload of (1*8 + 1.5*4 + 2*2.5) or 19 ops/output or 2.38 ops/input. Also not bad!

Three Half-Band Recursive Non-Linear Phase All-Pass Filters: 2, 3, and 4 Stages, 2.25 Ops/Output
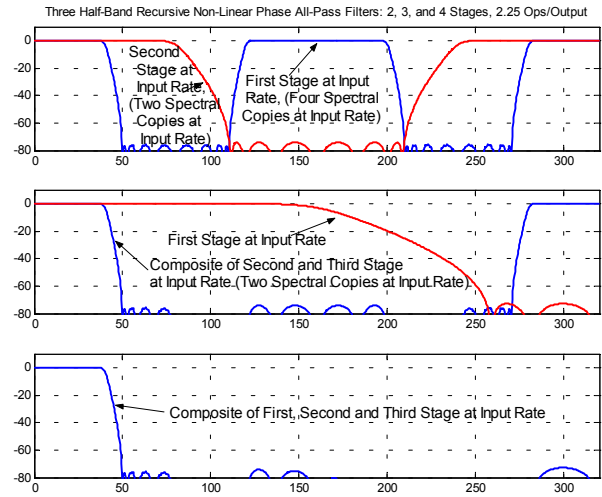
Figure 13. Spectral Response of Cascade Half-Band Non-Uniform Phase IIR Filters Referenced to Input Sample Rate

The spectral response of the IIR linear phase half-band filter Dyad is shown in figure 14.
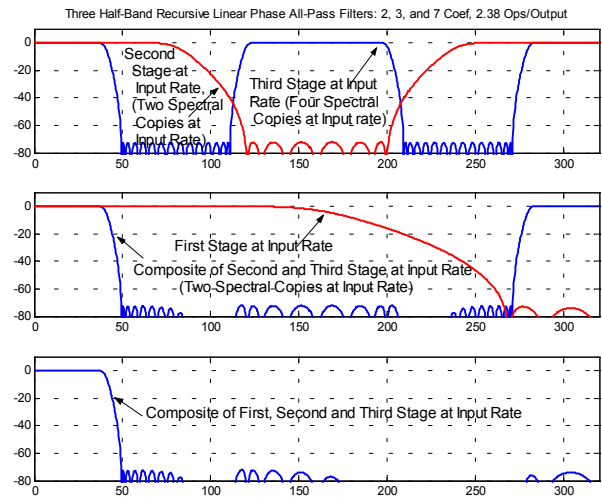
Three Half-Band Recursive Linear Phase All-Pass Filters: 2, 3, and 7 Coef, 2.38 Ops/Output

Figure 14. Spectral Response of Cascade Half-Band Uniform Phase IIR Filters Referenced to Input Sample Rate

5: SYSTEM COMPARISON

We presented a number of ways to reduce the bandwidth of an input signal with the option to also reduce the data rate commensurate with the bandwidth reduction. For the non-sample rate change options the filters we examined were standard IIR structures, standard FIR structures and structures formed by two path recursive all-pass filters. The conventional wisdom is that an IIR filter requires fewer operations than a FIR filter we saw this here where

the 7-th order Elliptic required 18-ops/output while the FIR required 104 ops/output. Interestingly, the two-path all-pass filter required 9-ops/output, a smaller number than the standard elliptic because the filter zeros were free in the all-pass structure. We will return to the non-down sampled option in a bit after we examine the down sample option.

When data rate is down sampled the conventional wisdom is faulty. The standard FIR filter can skip the computation of the non-retained output samples to realize the same workload per output but a reduced workload per input since the output work is amortized over the 8-inputs. The important parameter in a resample filter is the ratio N/M, which is ops/output normalized by inputs/output. Standard recursive filters must continue to compute the non-retained output samples because the filter needs these samples to compute the next output sample.

Embedding the resampling operation in the FIR filter lead to the polyphase structure, which significantly reduced the workload per input. A very efficient class of resampling filters can be formed when recursive all-pass filters are embedded in the polyphase structure. We illustrated a filter set exhibiting non-uniform phase and one exhibiting uniform phase. The polyphase version of the recursive all-pass filters showed marked improvement over the similar FIR partition. For comparison, the three 8-path options, FIR, non-uniform phase IIR, and Uniform phase IIR required 13, 2.7, and 5.3 ops/input.

Additional computational efficiency can be had when the resampling is embedded in a cascade of successive down sampling filters. The example we illustrated was a sequence of three half-band filters. For comparison, the three half-band options, FIR, non-uniform phase IIR, and Uniform phase IIR required 9.25, 2.25, and 2.7 ops/input.

We now return to the filtering task with the non-down sampling option. An interesting trick is to use one of the efficient down sample options, in particular either of the 8-path recursive all-pass or either of the cascade half-band recursive all-pass filters and then use the dual of same filter to up sample back to the original input sample rate. Using the filter for both down sampling and then up sampling doubles the workload. We note that doubling the work of any of the very efficient recursive all-pass filters is still more efficient by a factor of two to four than the best non re-sampled recursive filter. A graphical comparison of the relative workloads for the options we considered and discussed is shown in figure 15.

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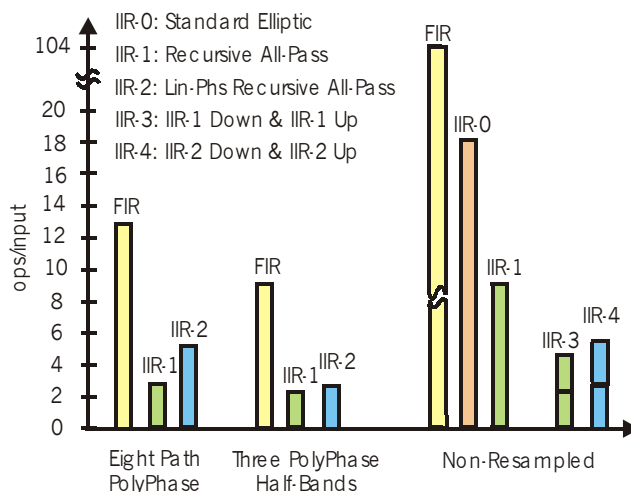
Figure-15 Workload Comparison for Different Filter Configurations to Reduce Bandwidth with 8-Stage Polyphase Filter, with 3-Stage Dyadic Half-band Filter, With Straight non-Resampling, and With Cascade 8-to-1 Down and 1-to-8 Up Sampling Filters

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